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A-Entropy for Generalized Observables

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We investigate A-entropy with respect to certain semispectral measures in a given state. It is shown that the entropy with respect to an observable describing "simultaneous" measurement of position and momentum is greater than the von Neumann entropy. Similar results are obtained for the fuzzy and sharp positions. The continuity properties of this entropy are also examined.

1. INTRODUCTION

The concept of A-entropy for spectrally absolutely continuous operators (for example position, momentum of particle in quantum mechanics) was introduced by Grabowski (1978). In that paper the possibility of defining such a notion for a semispectral measure was pointed out.

Here A-entropy with respect to certain semispectral measures [so-called generalized observables in sense (Davies and Lewis, 1970; Holevo, 1973)] will be investigated. In particular it is shown that entropy for a fuzzy observable of position is greater than for a sharp one. The continuity properties of this entropy are examined. Further it is shown that the entropy with respect to an observable describing "simultaneous" measurement of position and momentum in a given state is greater than the von Neumann entropy of this state. It is a certain continuous analogy of the following inequality (Klein, 1931; von Neumann, 1932)

$$-\sum_{n=1}^{\infty} \operatorname{Tr} \rho P_n \ln \operatorname{Tr} \rho P_n \geqslant -\operatorname{Tr} \rho \ln \rho$$

for an arbitrary state described by a density operator ρ and the resolution of identity $\{P_n\}_{n=1}^{\infty}$, $\sum_{n=1}^{\infty} P_n = 1$, where $H(\rho) = -\operatorname{Tr} \rho \ln \rho$ is the von Neumann entropy for the state ρ.

We recall the definition of an observable in the sense of Davies (1976).

Definition 1. Let Ω be a set with a σ-field of Borel subsets $\mathscr{F}$, and let $\mathscr{H}$ be a Hilbert space. A positive operator-valued measure on Ω is defined to be a map $A: \mathscr{F} \to B(\mathscr{H})$ [$B(\mathscr{H})$ is the algebra of bounded operators on $\mathscr{H}$] such that

(i) $A(E) \geqslant A(\varnothing) = 0$ for all $E \in \mathscr{F}$;

(ii) if $\{E_n\}$ is a countable collection of disjoint sets in $\mathscr{F}$, then $A(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} A(E_n)$, where the series converges in a weak operator topology. This measure is called an observable if also

(iii) $A(\Omega) = 1$.

For our purpose we assume that the Lebesgue measure m is defined on $\mathscr{F}$. Thus $(\Omega, \mathscr{F}, m)$ is a measure space.

The state as usual is described by a density operator (positive linear operator with a unit trace).

The probability that the measurement of A in the state ρ yields a result in the set $E \in \mathscr{F}$ is taken to be

$$p^A(E, \rho) = \operatorname{Tr} \rho A(E)$$

Let $p^A(\cdot, \rho)$ be absolutely continuous with respect to the Lebesgue measure (we call such observables *spectrally absolutely continuous*). Then there exists a nonnegative Radon–Nikodym derivative $(dp^A/dm)p^A(\cdot, \rho)$ with respect to m. Thus we propose the following definition.

Definition 2. Let A be a spectrally absolutely continuous observable. The expression

$$H(\rho, A) = -\int_{\Omega} \frac{dp^A}{dm} \ln \frac{dp^A}{dm}\, dm$$

is called A-entropy with respect to an observable A.

2. A-ENTROPY FOR THE FUZZY OBSERVABLES

Recently so-called fuzzy observables [Twareque Ali and Emch, 1974; Twareque Ali and Doebner, 1976; and Prugovecki, 1976 (and references therein)] have been investigated.

In usual quantum considerations a measurement is treated as an ideal one. No errors of observation are involved. In fact every physical measurement is subject to errors.

For example (Twareque Ali and Emch, 1974), a measurement of position of a particle can be made as precise as the applied instrument allows and it cannot be eliminated altogether. In this example—a particle moving on a real line—the Hilbert space corresponding to this system is $L^2(\mathbb{R})$ (a space of square integrable, complex-valued functions on $\mathbb{R}$), and the usual position operator is $(Q\psi)(x) = x\psi(x)$ for all $x \in \mathbb{R}$ and all $\psi(x) \in D(Q)$, where $D(Q)$ is the dense domain of Q in $L^2(\mathbb{R})$. The projector P corresponding to Q via the spectral theorem is defined by

$$(P(E)\psi)(x) = \chi_E(x)\psi(x)$$

where $\chi_E(x)$ is a characteristic function of E, and the probability that the particle is found in E when the system is in the state ψ is equal to $p_\psi(E) = (\psi, P(E)\psi)$. Let x_0 be the coordinate of the midpoint of the interval E. Assume that our apparatus has a finite resolution Δ and thus cannot distinguish between two points that are separated by a distance less than Δ. We may assume that the observed midpoint is distributed on $\mathbb{R}$ according to probability density $x \to f^\Delta(x)$ centered at x_0 and symmetric. Then

$$\tilde{p}_\psi(E) = \int \chi_E * f^\Delta(x)|\psi(x)|^2\, dx = (a(E)\psi, \psi)$$

and $(a(E)\psi)(x) = \chi_E * f^\Delta(x)\psi(x)$, where $*$ denotes the convolution. Obviously the measure $\tilde{p}_\psi(E)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Thus $H(\rho, a)$ for $\rho = |\psi\rangle\langle\psi|$ exists.

For such entropy the following theorem holds.

Theorem 1. Let $\mathscr{H} = L^2(\mathbb{R})$ and a is defined as before. Then for $\rho = |\psi\rangle\langle\psi|$ we have the inequality $H(\rho, a) \geqslant H(\rho, P)$.

Proof.

$$\begin{aligned} H(\rho, a) &= -\int |\psi|^2 * f^\Delta(y) \ln |\psi|^2 * f^\Delta(y)\, dy \\ &= -\iint |\psi(x)|^2 f^\Delta(y - x) \ln |\psi|^2 * f^\Delta(y)\, dx\, dy \qquad (1) \\ H(\rho, P) &= -\int |\psi(x)|^2 \ln |\psi(x)|^2\, dx \\ &= -\iint f^\Delta(y - x)|\psi(x)|^2 \ln |\psi(x)|^2\, dx\, dy \qquad (2) \end{aligned}$$

The difference between (1) and (2) is positive because $f^\Delta(x) \geqslant 0$ and $\int f^\Delta(x)\, dx = 1$.

This proof is a particular case of inequality for two probability densities $f(y)$ and $g(x)$ that are connected by the equality

$$g(x) = \int K(x, y) f(y)\, dy$$

where $K(x, y) \geqslant 0$ is a transition probability and $\int K(x, y)\, dx = \int K(x, y)\, dy = 1$ (Isihara, 1971).

This inequality agrees with our interpretation of $H(\rho, a)$ as an uncertainty of the outcome of measurement of an observable a in the state ρ. For the sharp measurement of position the uncertainty is less than for the fuzzy position. Our concept of entropy reflects the accuracy of measurement.

3. CONTINUITY PROPERTIES

The usual properties of A-entropy such as invariance properties, additivity, subadditivity, convexity, and so on were investigated by Grabowski (1978). In this section we examine only the continuity properties of A-entropy for a spectrally absolutely continuous observable. For this kind of entropy the result of Ochs (1976, Lemma 2) is applied. Thus $H(\rho, A)$ is not continuous with respect to the $\mathscr{L}^1$-topology, since in the case $m(\Omega) = \infty$ every neighborhood of dp^A/dm contains a probability density f with $H(f) = \infty$. However, for the observables that have spectral density satisfying certain conditions the weaker form of the continuity exists.

Definition 3. Observable A has a spectral density if for each $\lambda \in \Omega$ there exists a bounded positive definite operator $F(\lambda)$ $[F(\lambda) \in B(\mathscr{H})]$ satisfying

$$A(E) = \int_E F(\lambda)\, dm(\lambda), \quad \text{for } E \in \mathscr{F} \tag{3}$$

in a weak sense.

For such observables the following dominated convergence theorem may be proved.

Theorem 2. Let $B \geqslant 0$ be an operator such that $H(B, A) < \infty$. Let ρ_n be a sequence of density operators converging weakly toward a density operator ρ with $\rho_n \leqslant \rho + B$ and $F^2(\lambda) = F(\lambda)$. Then

$$H(\rho_n, A) \to H(\rho, A)$$

Owing to the condition $F^2(\lambda) = F(\lambda)$, we have the continuous resolution of identity.

Proof.

$$\rho_n \leqslant B + \rho, \qquad F(\lambda)\rho_n F(\lambda) \leqslant F(\lambda)\rho F(\lambda) + F(\lambda)BF(\lambda)$$

$$\rho_n \xrightarrow{\mathrm{w}} \rho \Rightarrow |\mathrm{Tr}\, \rho F(\lambda) - \mathrm{Tr}\, \rho_n F(\lambda)| \leqslant \|F(\lambda)\|\, \|\rho - \rho_n\|_1$$

On the set of states the weak convergence implies a convergence in a trace norm (Davies, 1976). Thus Theorem 3 of (Ochs, 1976) may be applied.

When $\|F(\lambda)\| \leqslant 1$, $H(\rho, A)$ is lower semicontinuous in the trace norm.

Theorem 3. For an observable satisfying (3), and when $\|F(\lambda)\| \leqslant 1$ and Ω is n-dimensional, the A-entropy is lower semicontinuous on the set of states with respect to the topology induced by the trace norm.

Proof. Let $\mathscr{C}(\Omega)$ be a family of bounded subsets. Then for $E \in \mathscr{C}(\Omega)$ the functional

$$h(\rho, A, E) = -\int_E \mathrm{Tr}\, \rho F(\lambda) \ln \mathrm{Tr}\, \rho F(\lambda)\, dm(\lambda)$$

is continuous in the trace norm. We take a sequence $E_i \to \Omega$, where $E_i \in \mathscr{C}(\Omega)$. Then $h(\rho, A, E_i) \to H(\rho, A)$, owing to $\|F(\lambda)\| \leqslant 1$, is a limit of nondecreasing sequence of continuous functions and hence is lower semicontinuous.

For example, for an observable describing a fuzzy measurement the spectral density is $(F_y\psi)(x) = f^{\Delta}(x - y)\psi(x)$. If f^{Δ} is essentially bounded, $f^{\Delta} \in L^{\infty}(\mathbb{R})$ and such that $\|f^{\Delta}\|_{\infty} \leqslant 1$, then A-entropy for that observable is lower semicontinuous. Physically interesting A-entropy that satisfies these conditions is presented in the next section.

4. "SIMULTANEOUS" MEASUREMENTS AND A-ENTROPY

There are some measurements whose result can be regarded as a pair of real numbers or as a single complex number. An optimal "simultaneous" measurement of position Q and momentum P which do not commute is an example. We can introduce operators a and a^+ satisfying the commutation relation $[a, a^+] \subseteq I$ and such that

$$a = (Q + iP)/\sqrt{2}, \qquad a^+ = (Q - iP)/\sqrt{2}$$

a has eigenstates $|z\rangle : a|z\rangle = z|z\rangle$ (coherent states). The observable corresponding to such measurement is

$$A(E) = \frac{1}{\pi}\int_E |z\rangle\langle z| d^2z$$

in a weak sense and $E \in \mathscr{B}(\mathbb{C})$, $\mathscr{B}(\mathbb{C})$ is a σ-algebra of Borel subsets of the complex plane. $A(E)$ has the spectral density $P_z = |z\rangle\langle z|$. $\operatorname{Tr} \rho A(E)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}$, i.e., d^2z/π.

Then the uncertainty of the simultaneous measurement of P and Q in the state ρ is

$$H(\rho, A) = -\frac{1}{\pi}\int_{\mathbb{C}} d^2z\langle z|\rho|z\rangle \ln \langle z|\rho|z\rangle$$

Entropy $H(\rho, A)$ has a certain connection with classical entropy.[1] Usually in statistical mechanics a classical entropy of probability density $\rho(q, p)$ on the phase space Γ is described in the following way

$$H_{cl}(\rho) = -\frac{1}{h}\int_{\Gamma} \rho(q, p) \ln \rho(q, p)\, dq\, dp$$

where h is the Planck constant (we consider the one-dimensional case). This expression has a semiclassical character owing to h which is the smallest volume in the phase space of one particle. The appearance of this smallest volume is a consequence of the uncertainty relation. The classical entropy

[1] This interpretation of $H(\rho, A)$ was pointed out to me by A. Wehrl.

may be negative. This fact is very difficult to interpret and is a result of a localization of $\rho(q, p)$ on the region smaller than h. Of course it is in contradiction with the quantum Heisenberg relation. Let $H_{cl} \leqslant 0$. Then $\rho(q, p) \geqslant 1$ on a certain subset $X \subset \Gamma$ with the Liouville measure Δ. Because

$$\frac{1}{h}\int_{\Gamma} \rho(q, p)\, dq\, dp = 1$$

we have

$$\frac{1}{h}\int_{\Gamma} \rho(q, p)\, dq\, dp = \frac{1}{h}\int_{X} \rho(q, p)\, dq\, dp + \frac{1}{h}\int_{\Gamma - X} \rho(q, p)\, dq\, dp$$

$$\geqslant \frac{\Delta}{h} + \frac{1}{h}\int_{\Gamma - X} \rho(q, p)\, dq\, dp \geqslant \frac{\Delta}{h}$$

Thus $\Delta < h$.

Our entropy $H(\rho, A)$ has a very nice property. The range of this entropy is equal to extended $\mathbb{R}^+$. It easily follows from the inequality

$$0 \leqslant \langle z|\rho|z\rangle \leqslant \|\rho\|_1 \|P_z\| = 1$$

Because of this property, $H(\rho, A)$ may be interpreted as a classical entropy, where the probability density $\rho(z) = \rho(q, p) = \langle z|\rho|z\rangle$ corresponds to the density operator ρ.

For this entropy the following inequality is true and shows the connection between $H(\rho, A)$ and the von Neumann entropy.

Theorem 4. Let $A(E)$ be as before and let ρ be a density operator. Then

$$H(\rho, A) > H(\rho) \tag{4}$$

where $H(\rho) = -\mathrm{Tr}\, \rho \ln \rho$ is the von Neumann entropy.

Proof. Using the spectral decomposition of ρ, $\rho = \sum_{i=1}^{\infty} p_i P_{\psi_i}$, $\mathrm{Tr}\, P_{\psi_i} = 1$, $1 \geqslant p_i \geqslant 0$, we have

$$H(\rho, A) = -\int_{\mathbb{C}} d^2z \frac{1}{\pi} \sum_{i=1}^{\infty} p_i |\langle z|\psi_i\rangle|^2 \ln \sum_{i=1}^{\infty} p_i |\langle z|\psi_i\rangle|^2$$

From the normalization conditions

$$\frac{1}{\pi}\int |\langle z|\psi_i\rangle|^2 d^2z = 1 \quad \text{and} \quad \sum_{i=1}^{\infty} |\langle z|\psi_i\rangle|^2 = 1$$

and from the arguments in Theorem 1 we have the desired inequality. Equality would hold for $p_i = \langle z|\rho|z\rangle$, which is impossible because of hermiticity of ρ and overcompleteness of coherent states.

This inequality is a certain continuous analogue of the inequality mentioned in the Introduction. In fact, using the base proposed by von Neumann for $z = \pi^{-1/2}(l + im)$, where l, m are integers (here $P_z = |z\rangle\langle z|$ is a discrete resolution of identity), we have the following inequality

$$-\sum_{l,m=0}^{\infty} \operatorname{Tr} \rho P_{l,m} \ln \operatorname{Tr} \rho P_{l,m} \geqslant -\operatorname{Tr} \rho \ln \rho \tag{5}$$

Remarks. Let ρ be a classical state (Davies, 1976), i.e., it has the so-called P-representation

$$\rho = \frac{1}{\pi}\int d^2z P(z)|z\rangle\langle z|, \qquad P(z) \geqslant 0, \qquad \frac{1}{\pi}\int P(z)\, d^2z = 1$$

On the set of such states the following inequalities hold.

Property 1. $p_i = \operatorname{Tr} \rho P_i = (1/\pi)\int P(z)\langle z|P_i|z\rangle d^2z$, and the preceding arguments give

$$-\operatorname{Tr} \rho \ln \rho \geqslant -\frac{1}{\pi}\int P(z) \ln P(z) d^2z \tag{6}$$

Property 2.

$$H(\rho, A) \geqslant 1 \tag{7}$$

$\langle z|\rho|z\rangle = P * f_G(z)$, $f_G(z) = (1/\pi)e^{-|z|^2}$. Changing variables in the integral, treating $P(z)$ as the transition probability and again using the same arguments, we obtain (7).

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After the first version of this paper was prepared, I learned that A. Wehrl (Vienna) proved inequalities (4) and (6) using different methods ("On the relation between classical and quantum-mechanical entropy," preprint Wien). Wehrl conjectured that (7) holds for all states. Recently E. Lieb proved this conjecture (private communication from A. Wehrl).

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